

SEMI-CLASSICAL RESONANCES ASSOCIATED WITH A PERIODIC ORBIT OF HYPERBOLIC TYPE

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Abstract. We consider in this Note resonances for a h -Pseudo-Differential Operator $H(x, hD_x; h)$ on $L^2(M)$ induced by a periodic orbit of hyperbolic type, as arises for Schrödinger operator with AC Stark effect when $M = \mathbf{R}^n$, or the geodesic flow on an axially symmetric manifold M , extending Poincaré example of Lagrangian systems with 2 degrees of freedom. We generalize the framework of [GéSj], in the sense that we allow for hyperbolic and elliptic eigenvalues of Poincaré map, and look for so-called semi-excited resonances with imaginary part of magnitude $-h \log h$, or h^s , with $0 < s < 1$.

I. Hypotheses and the main result.

For simplicity, we present our results with $M = \mathbf{R}^n$. Let $H(x, hD_x; h)$ be a self-adjoint h -PDO on $L^2(\mathbf{R}^n)$

$$(1.1) \quad H^w(y, hD_y; h)u(y; h) = (2\pi h)^{-n} \int \int e^{i(y-y')\eta'/h} H\left(\frac{y+y'}{2}, \eta'; h\right) u(y') dy' d\eta'$$

We assume it has Weyl symbol $H(y, \eta; h) \in S^0(m)$, where m is an order function (for example $m(y, \eta) = (1 + |\eta|^2)^M$), and

$$S^N(m) = \{H \in C^\infty(T^*\mathbf{R}^n) : \forall \alpha \in \mathbf{N}^{2n}, \exists C_\alpha > 0, |\partial_{(y,\eta)}^\alpha H(y, \eta; h)| \leq C_\alpha h^N m(y, \eta)\}$$

with the semi-classical expansion $H(y, \eta; h) \sim H_0(y, \eta) + hH_1(y, \eta) + \dots, h \rightarrow 0$. Here H_0 is the principal symbol of H , H_1 its sub-principal symbol. We assume that $H(y, \eta; h)$ is elliptic (i.e. $H + i$ elliptic) and defines an analytic symbol in the sense of [Sj] in a sector

$$\Gamma_0 = \{(y, \eta) \in T^*\mathbf{C}^n : |\operatorname{Im}(y, \eta)| \leq \operatorname{const} \cdot \langle \operatorname{Re}(y, \eta) \rangle\}$$

Let the energy surface $H_0^{-1}(E_0)$ be regular for some $E_0 \in \mathbf{R}$, that we may set up to 0. So the Hamiltonian vector field X_{H_0} has no fixed point on $H_0^{-1}(0)$, hence on nearby energy surfaces $H_0^{-1}(0)$. Let $\Phi^t = \exp(tX_{H_0}) : T^*\mathbf{R}^n \rightarrow T^*\mathbf{R}^n$ and

$$(1.2) \quad K_E = \{\rho \in T^*\mathbf{R}^n, H_0(\rho) = E, \Phi^t(\rho) \text{ doesn't grow to infinity as } |t| \rightarrow \infty\}$$

be the trapped set at energy E . We assume that $K_0 = \gamma_0$ is a periodic orbit of period T_0 . Let \mathcal{P}_0 be Poincaré map (first return map), acting on a Poincaré section $\Sigma(\rho) \subset T^*\mathbf{R}^n, \rho \in \gamma_0$. Assume also that 1 is not an eigenvalue of $d\mathcal{P}_0|_{\gamma_0}$, then $\Sigma(\rho)$ is transverse to the center manifold $\bar{\gamma}$, identified with a

neighborhood of the zero-section in $T^*\mathbf{S}^1$. Each $\Sigma(\rho)$, $\rho \in \gamma_0$, identifies with $\Sigma \approx T^*\mathbf{R}^d$ (locally along $\bar{\gamma}$, modulo the action of Hamiltonian flow). Both $\bar{\gamma}$ and Σ are symplectic manifolds, and for small E : $K_E = \gamma_E$ is a periodic orbit of period T_E , $\bigcup_E \gamma_E = \bar{\gamma}$. For $\rho \in \gamma_0$, let $\lambda_j, 1 \leq j \leq 2d = 2(n-1)$ be the eigenvalues of $A_0(\rho) = d\mathcal{P}_0(\rho) : \mathbf{C}^{2d} \rightarrow \mathbf{C}^{2d}$ (Floquet multipliers). The space \mathbf{C}^{2d} has the orthogonal symplectic decomposition in (generalized) eigenspaces F_λ relative to the family $(\lambda_j)_{1 \leq j \leq 2d}$. We are interested in the case where $A_0(\rho)$ is partially hyperbolic, i.e. has at least one eigenvalue λ of modulus $\neq 1$. Assume also that Poincaré map is non degenerate, i.e. $F_{\pm 1} = \{0\}$, and also $F_\lambda = \{0\}$ for all $\lambda \leq 0$. We say that $\lambda \in \mathbf{C}$ is *elliptic* (ee for short) if $|\lambda| = 1$ ($\lambda \neq \pm 1$) and *hyperbolic* (he) if $|\lambda| \neq 1$; if moreover $\lambda \in \mathbf{R}$ we call it *real hyperbolic* (hr) and *complex-hyperbolic* (hc) otherwise. Under the last assumption we can define $B = \log A$. Eigenvalues $\mu = \mu(\lambda) = \log \lambda$ of B (Floquet exponents) verify $\mu(\bar{\lambda}) = \overline{\mu(\lambda)}$. Accordingly, exponent μ is said ee if $\text{Re } \mu = 0$, hr if $\mu \in \mathbf{R} \setminus 0$, and hc if $\mu \in \mathbf{C} \setminus \mathbf{R}$. So eigenvalues of B have the form $\mu_j, -\mu_j, \bar{\mu}_j, -\bar{\mu}_j \neq 0$, $\text{Re } \mu_j \geq 0$, with same multiplicity. Let $b(\rho) = \frac{1}{2}\sigma(\rho, B\rho)$ (Hermitian form), and r be the number of distinct μ_j 's. For simplicity, assume $r = d$, hence b diagonalizable. We know [Bry] that in a suitable basis $b(\rho)$ is a linear combinaison of elementary quadratic polynomials Q_j . If $\mu_j \in i\mathbf{R}$ (elliptic sector), we choose in $F_{\lambda_j} \oplus F_{-\lambda_j}$ symplectic coordinates (“harmonic oscillator coordinates”) such that $Q_j = \frac{1}{2}(x_j^2 + \xi_j^2)$. If $\mu_j \in \mathbf{R}$ (real hyperbolic sector), one has $Q_j = x_j \xi_j$, while in the complex-hyperbolic sectors, where $\mu_j = c_j + id_j$, one has $Q_j(x, \xi) = c_j Q'_j(x, \xi) - d_j Q''_j(x, \xi)$, with $Q'_j(x, \xi) = x_{2j-1} \xi_{2j-1} + x_{2j} \xi_{2j}$, $Q''_j(x, \xi) = x_{2j-1} \xi_{2j} + x_{2j} \xi_{2j-1}$. In suitable complex symplectic coordinates, Q_j has always the form $Q_j = x_j \xi_j$. The Q_j 's play an important rôle, since they are formally “transverse eigenvectors” for H , microlocalized near γ_0 .

Our next Hypothesis is relative to *partial hyperbolicity* of Poincaré map, in the sense that there exists $j \in \{1, \dots, r\}$, such that $\text{Re } \mu_j > 0$. For hyperbolic dynamic systems, we know [A] that generically only one μ_j has $\text{Re } \mu_j > 0$. Let F_{μ_j} , $\text{Re } \mu_j \geq 0$ denote again the eigenspace associated with μ_j . We can rewrite the decomposition of \mathbf{C}^{2d} in the sum of unstable space F^+ and stable space F^- :

$$F^+ = \bigoplus_{j=1}^{r=d} F_{\mu_j}, \quad F^- = \bigoplus_{j=1}^{r=d} F_{-\mu_j}$$

where $F^\pm \simeq \mathbf{C}^d$ are (complex) Lagrangian subspaces of \mathbf{C}^{2d} invariant under the flow of X_b , and such that if there exists an elliptic element ($\text{Re } \mu_k = 0$), then for small $\theta > 0$, $e^{-i\theta} X_b$ is “expansive” on F^+ , “contractive” on F^- . We call the elliptic element with positive imaginary part an “eigenvalue of the first kind”. Elliptic elements contribute to the center manifold. So generally we need consider Hamiltonian flow for complex times, which is achieved in the framework of complex eigenvalues (resonances).

Our last Hypothesis concerns the non-resonance condition relative to Floquet exponents (see [Br]), which is required to achieve Birkhoff normal form, namely

$$(1.7) \quad r = d \text{ and } \forall k_1, \dots, k_r \in \mathbf{Z} : \sum_{j=1}^r k_j \mu_j \in 2i\pi\mathbf{Z} \implies \sum_{j=1}^r k_j \mu_j = 0$$

For instance, when $n = 2$ and $\mu_1 = i\omega_1$, it takes the form $k_1 i\omega_1 \in 2i\pi\mathbf{Z}$, iff $k_1\omega_1 = 0$, i.e. the rotation number ω_1 is irrational. We need also the strong non-resonance condition on Floquet exponents:

$$(1.8) \quad r = d \text{ and } \forall k_1, \dots, k_r \in \mathbf{Z} : \sum_{j=1}^r k_j \mu_j \in 2i\pi\mathbf{Z} \implies k_j = 0, j = 1, \dots, d$$

b) *Examples:*

1) The Model Hamiltonian

$$(1.9) \quad H_{\text{mod}}(hD_t, x, hD_x; h) = -hD_t + \sum_{j=1}^d \mu_j Q_j^w(x, hD_x)$$

$Q_j^w(x, hD_x) = \frac{1}{2}(x_j hD_{x_j} + hD_{x_j} x_j)$, with Periodic Boundary Conditions on $\mathbf{S}^1 \times \mathbf{R}^d$, serves as a guideline throughout this work. Here x may denote complex variables, in some Bargmann representation of the Hamiltonian.

2) A physical example is given by $H(y, hD_y) = -h^2 \Delta_y + |y|^{-1} + ay_1$ on \mathbf{R}^n (repulsive Coulomb potential perturbed by Stark effect) near an energy level $E > 2/\sqrt{a}$. It can be generalized in the case of 3 bumps of potential (“Monkey Saddle”), see [Sj3].

3) The geodesic flow on the one-sheeted hyperboloid in \mathbf{R}^3 has an (unstable) periodic orbit of hyperbolic type (Poincaré example). This example generalises [Chr, App.C] to a surface of revolution in \mathbf{R}^4 , involving one pair of real-hyperbolic elements and two pairs of elliptic elements. Our method easily carries to the case when $H(y, hD_y)$ is the geodesic flow on such manifolds.

Before stating our result we recall the index of a symplectic arc (Gelfand-Lidskiy, or Conley-Zehnder index), which appears in the quantization condition when elliptic elements occur. Standard Maslov index is associated with a differentiable loop in $\text{Sp}(2n; \mathbf{R})$, while Conley-Zehnder index (in the formulation of [SaZe]) is defined for a differentiable path $\Psi : [0, T] \rightarrow \text{Sp}(2n; \mathbf{R})$ such that $\Psi(0) = \text{Id}$ and $\det(\text{Id} - \Psi(T)) \neq 0$ for some $T' \in]0, T[$. In the present case, let $Y(t)$ solve the variational system along γ_E , i.e. $\dot{Y}(t) = JH''(\Phi^t(\rho))Y(t)$, $Y(0) = \text{Id}$, where we recall $\Phi^t(\rho)$ is the flow of X_{H_0} issued from $\rho \in \gamma_E$ as in (1.2). We define $\Psi(t) = d\mathcal{P}_E(t)$ as the co-restriction of $Y(t)$ to Poincaré sections, i.e. $\Psi : [0, T] \rightarrow \text{Sp}(2d; \mathbf{R})$; here T is taken to be the period of γ_E . Then, Conley-Zehnder index can be interpreted as the mean winding number of the eigenvalues of the first kind, and is computed here most easily using Birkhoff normal form along γ_E .

We are concerned with semi-classical *resonances* of H near $E_0 = 0$, in the framework of “complex scaling” theory and its extensions [ReSi], [HeSj], i.e. the discrete spectrum of some suitable analytic continuation of H as a closed, Fredholm, but non-selfadjoint operator. Our main result, the generalized Bohr-Sommerfeld quantization condition, can be formulated as follows:

Theorem: *Under the hypotheses above, let (after re-ordering) $\mu_j = i\omega_j$, $j = 1, \dots, \ell$, $\omega_j > 0$ be the set of elliptic Floquet exponents for H_0 . Recall H_1 from (1), and let $H_1(x(t), \xi(t)) dt$ the sub-principal 1-form. We define the semi-classical action along γ_E , mod $\mathcal{O}(h^2)$, by $\mathcal{S}(E; h) = S_0(E) + hS_1(E) + \mathcal{O}(h^2)$, with*

$$(1.10) \quad S_0(E) = \frac{1}{2\pi h} \int_{\gamma_E} \xi dx$$

$$(1.11) \quad S_1(E) = -\frac{1}{2\pi} \int_0^{T(E)} H_1(x(t), \xi(t)) dt + \frac{1}{2\pi} (\omega_1(E) + \cdots + \omega_\ell(E)) + \frac{g_\ell}{2}$$

where $\omega_1(E), \dots, \omega_\ell(E)$ are the actions along (complex) arcs on a Poincaré section, $\omega_j(E) = \omega_j + \mathcal{O}(E)$, and $g_\ell \in \mathbf{Z}$ Cohnley-Zehnder index of γ_E . Then the resonances of H near 0 are given (at first order in h) by generalized Bohr-Sommerfeld (BS) quantization condition

$$(1.12) \quad \frac{1}{2\pi h} \mathcal{S}(E; h) + \frac{1}{2i\pi h} \sum_{j=1}^d k_j \mu_j(E) = m \in \mathbf{Z}, \quad k = (k_1, \dots, k_d) \in \mathbf{Z}^d$$

provided $|m|h \leq \varepsilon_0$, $|k|h \leq h^\delta$, with $0 < \delta < 1$.

In the elliptic case, a similar theorem (for real spectrum) was obtained in [Ba], [BaLa], and [Ra]; in the real hyperbolic case, in [GéSj] for $|\operatorname{Im} E| = \mathcal{O}(h)$, and [Sj4] in dimension 2 with $|\operatorname{Im} E| = \mathcal{O}(h^\delta)$ or even $|\operatorname{Im} E|$ small enough independently of h , but selecting a single Floquet parameter in the semi-classical Floquet decomposition of H near γ_0 , i.e. few “longitudinal” or “principal” quantum numbers $m \in \mathbf{Z}$. For related results about trace formulas or concentration of eigenvalues, see [Vo], [SjZw], [NoSjZw], [Chr]. For the wave equation outside convex obstacles, see [Ik], [Gé].

II. Outline of proof.

The main object to be constructed is the semi-classical *monodromy operator* $M^*(E)$, a h -FIO quantizing Floquet operator associated with the periodic orbit.

1) Birkhoff normal form (BNF)

We start to find suitable coordinates near $\bar{\gamma}$. When $\operatorname{Re} \mu_j > 0$ for all $1 \leq j \leq d$, the stable/unstable manifold theorem guarantees the existence of involutive manifolds Γ_\pm in a neighborhood of γ_0 with

$$(2.1) \quad T_\rho \Gamma_+ + T_\rho \Gamma_- + T_\rho \bar{\gamma} = T^* \mathbf{R}^n, \quad \rho \in \gamma_0$$

There are (real) symplectic coordinates (t, τ, x, ξ) such that $\xi = 0$, $d\xi \neq 0$ on Γ_+ , $x = 0$, $dx \neq 0$ on Γ_- , and (t, τ) parametrize $\bar{\gamma}$. Write here H instead of H_0 . Intersecting with the energy surfaces $H^{-1}(E)$ gives the foliation

$$(2.2) \quad T_\rho \Gamma_+(E) + T_\rho \Gamma_-(E) + T_\rho \gamma(E) = T_\rho^* H^{-1}(E), \quad \rho \in \gamma(E)$$

and $\Gamma_\pm(E)$ are Lagrangian submanifolds. In these coordinates $H(y, \eta) = f(\tau) + \langle B(t, \tau, x, \xi)x, \xi \rangle$. Here f parametrizes the energy parameter $f(\tau) = E$, and is related with the period $T(E)$ of $\gamma(E)$ by $f'(\tau) = \frac{2\pi}{T \circ f(\tau)}$. Performing a first canonical transformation gives $B(t, \tau, x, \xi) = B_0(\tau) + \mathcal{O}(|\tau|, |x, \xi|)$, where the eigenvalues of $B_0(\tau)$ are Floquet exponents for \mathcal{P}_E with positive real part. When $\operatorname{Re} \mu_k = 0$ for some k , (2.1) and (2.2) still hold provided we take complex variables. In both cases however, under the non resonance conditions (1.7), (1.8) BNF holds in the classical sense [Bry] as well as in the semi-classical sense [GuPa] and takes, modulo a small remainder term, operator $H^w(y, hD_y; h)$ to a polynomial in hD_t and $Q_j^w(x, hD_x)$, $Q_j(x, \xi)$ being one of the quadratic polynomials above. In

particular, the principal part of $H^w(y, hD_y; h)$ in BNF is given by (1.9) in some suitable Bargmann (still formal) representation, provided a reparametrization of energy.

2) Microlocalisation in the complex domain

As trying to construct quasi-modes for $H^w(y, hD_y; h)$ microlocalized near the stable/unstable manifolds Γ_{\pm} one meets the difficulty, already observed in [Du], that because of hyperbolic elements, there is no smooth X_{H_0} -invariant density on Γ_{\pm} . On the other hand, elliptic elements are responsible for caustics in the time-evolution (or Cauchy problem). These difficulties naturally disappear in the framework of resonances, provided we are working within the framework of h -FIO's with complex phase of positive imaginary part, as in [MeSj], [Sj], see also [M]. Resonances here are considered from the point of view of analytic dilations and Lagrangian deformations; taking into account that there exists an *escape function* (that grows along the flow of X_{H_0}) which implies kind of a “virial condition” outside the trapped set γ_0 , the most relevant region of phase-space for such deformations is a neighborhood of γ_0 . Here we make a complex scaling of the form $(x, \xi) \mapsto (e^{i\theta}x, e^{-i\theta}\xi)$, followed also by a small deformation in the (t, τ) variables. Our main tool is FBI transformation (metaplectic FIO with complex phase) which takes the form, in coordinates $(s, y; t, x) \in T^*\mathbf{R}^n \times T^*\mathbf{C}^n$ adapted to Γ_{\pm} as in BNF

$$T_0 u(x, h) = \int e^{i\varphi_0(t, s; x, y)/h} u(s, y) ds dy, \quad u \in L^2(\mathbf{R}^n)$$

where $\varphi_0(t, s; x, y) = \varphi_1(t, s) + \varphi_2(x, y)$, $\varphi_1(t, s) = \frac{i}{2}(t-s)^2$, $\varphi_2(x, y) = \frac{i}{2}[(x-y)^2 - \frac{1}{2}x^2]$. The corresponding canonical transformation is $\kappa_0 = (\kappa_1, \kappa_2)$, with $\kappa_1 : (s, -\partial_s \varphi_1) \mapsto (t, \partial_t \varphi_1)$, $\kappa_2 : (y, -\partial_y \varphi_2) \mapsto (x, \partial_x \varphi_2)$, and the corresponding pl.s.h. weight $\Phi_0 = \Phi_1 + \Phi_2$, with

$$\Phi_1(t) = \sup_{s \in \mathbf{R}} (-\operatorname{Im} \varphi_1(t, s)) = (\operatorname{Im} t)^2/2, \quad \Phi_2(x) = \sup_{y \in \mathbf{R}^d} (-\operatorname{Im} \varphi_2(x, y)) = |x|^2/4$$

In a very small neighborhood of γ_0 , whose size will eventually depend on h , corresponding to $\theta = -\pi/4$, and that we call the “phase of inflation”, it turns out that $H^w(y, hD_y; h)$ takes the simple form above whose principal term is given in (1.9), and the corresponding weight $\tilde{\Phi}(t, x)$ is just $\Phi_0(t, x)$. Otherwise we take θ small enough in a somewhat larger neighborhood of γ_0 , which we call the “linear phase”. These weights are deformed continuously in phase-space, depending on the escape function, and patched together in overlapping regions, so to define a globally pl.s.h. function in complex x -space. They also define the contour integral for writing a h -FIO in the complex domain; when the weight is quadratic, and in the particular case of a h -PDO with C^∞ amplitude a

$$Av(z, h) = \int_{\Gamma(z)} e^{i(z-y)\eta/h} a((z+y)/2, \theta, h) v(y) dy \wedge d\eta$$

we have $\Gamma(z) = \{\eta = \frac{2}{i} \frac{\partial \Phi}{\partial z}(\frac{z+y}{2}); y \in \mathbf{C}^n\}$. Where a is not analytic, due in particular to BNF, a denotes an almost analytic extension. When the weight is not quadratic, they still define “good contours” in the sense of [Sj].

3) Poisson operator and its adjoint

We look for $K(t, E) : L^2(\mathbf{R}^d) \mapsto L^2(\mathbf{R}^d)$, microlocalized near $\Gamma_+(E)$, such that

$$H(hD_t, x, hD_x; h)K(t, E) = 0, \quad K(0, E) = \text{Id}$$

In the phase of inflation, this is an OIF $H_{\Phi_2}(\mathbf{C}^d) \rightarrow H_{\Phi_2^t}(\mathbf{C}^d)$

$$K(t, E)v(x; h) = \int \int e^{i(S(t, x, \eta) - y\eta)/h} a(t, x, \eta; E, h)v(y) dy \wedge d\eta$$

where $\Phi_2^t(x) = \Phi_2 \circ \kappa_t$ (where κ_t denotes the flow of X_{H_0} in complex coordinates, when restricted to (x, ξ) variables), and the integration is carried over a suitable contour as above. We solve eikonal and transport equations for S , with initial condition $S(0, x, E) = x\eta$, and find for instance $S = S_0 + \tilde{S}$, where

$$S_0(t, x; \eta) = -Et + \sum_{j=1}^d x_j \eta_j e^{\mu_j t}$$

is the phase for the model Hamiltonian, and the correction $\tilde{S}(t, x, E)$ has the asymptotic form

$$\tilde{S}(t, x, E) = -t \sum_{k \geq 2} s_k E^k + \sum_{\alpha, l, m} c_{\alpha, l, m} E^\alpha Q^\ell(x, \eta) e^{\langle m, \mu \rangle t}$$

with $|\ell| + \alpha \geq \sup(|m|, 2)$. So $K(t, E)$ is in BNF, i.e. its kernel depends on (x, η) only through $Q_j(x, \eta)$. Its adjoint $K^*(E) = \int_{\gamma^*(E)} K^*(t, E) dt$ is continuous $\int^\oplus H_{\Phi_2^t}(\mathbf{C}^d) \rightarrow H_{\Phi_2}(\mathbf{C}^d)$ provided $\gamma^*(E) = \{|\text{Im } t| = \varepsilon\}$, $0 < \varepsilon < 2|E + \sum_{k \geq 2} s_k E^k|$. This fixes the relative sizes between τ and $|x, \xi|^2$. In the model case we have simply

$$K(t, E)v(x, h) = (2\pi h)^{-d} \int \int e^{i(-Et + \sum x_j \eta_j e^{\mu_j t} - y\eta)/h} e^{(\sum \mu_j)t/2} v(y) dy d\eta$$

4) Normalization.

We proceed first formally, i.e. with operators in the real domain. We use the “flux norm” of [SjZw] to normalize Poisson operator. Let $\chi \in C^\infty(\mathbf{R})$, be equal to 0 near 0, 1 near $[2\pi, \infty[$. Pseudo-differential calculus shows that there is a h -PDO $B(E) = B^w(x, hD_x; E)$ such that $L(t, E) = K(t, E)B(E)$ satisfies

$$(2.3) \quad \left(\frac{i}{h}[H, \chi(t)]L(t, E)v\right) = (v|v)$$

that is $\int dt L(t, E)^* \frac{i}{h}[H, \chi(t)]L(t, E) = \text{Id}_{L^2(\mathbf{R}^d)}$. In the other way, we need also consider operators like $\int L(s, E)L^*(E)dE$, or $\int dE L^*(E)L(s, E)$. In the model case

$$\int dE L(s, E)L^*(E)u(s, x) = \int \int e^{i(-E(s-t) + (x-y')\eta')/h} e^{\tilde{\mu}(s-t)/2} u(t, y' e^{\mu(s-t)}) dy' d\eta' dt dE = u(s, x)$$

by asymptotic Fourier inversion formula, since $e^{\tilde{\mu}(s-t)/2} u(t, y' e^{\mu(s-t)})$ is independent of (E, η') . In the general case, we only have $\int dE L(\cdot, E)L^*(E) = \text{Id}_{L^2(\mathbf{R}^n)} + \mathcal{O}(h)$, and similarly $\int dE L^*(E)L(s, E) =$

$\text{Id}_{L^2(\mathbf{R}_s^n)} + \mathcal{O}(h)$, denoting by \mathbf{R}_s^n the section $\{s\} \times \mathbf{R}^d$ of \mathbf{R}^n (in BNF coordinates). By Pseudo-differential calculus there exists $P(s, E) = \text{Id}_{L^2(\mathbf{R}_s^n)} + \mathcal{O}(h)$ such that

$$(2.4) \quad \int dE L(\cdot, E) P(\cdot, E) L^*(E) = \text{Id}_{L^2(\mathbf{R}^n)}, \quad \int dE L^*(E) L(s, E) P(s, E) = \text{Id}_{L^2(\mathbf{R}_s^n)}$$

These (formal) computations can be carried out in the framework of FIO's in the complex domain. A similar situation was met in [BdMSj] when considering Bergman and Szegő projectors.

5) *The monodromy operator.*

We set $K_0(t, E) = K(t, E)$ where $K(t, E)$ is Poisson operator with Cauchy data at $t = 0$, and $L_0(t, E) = K_0(t, E)B(E)$; we set similarly $L_{2\pi}(t, E) = K_0(t - 2\pi, E)B(E)$. The monodromy operator (or semi-classical Poincaré map) is defined by

$$(2.5) \quad M^*(E) = L_{2\pi}^*(E) \frac{i}{h} [H, \chi] L_0(\cdot, E)$$

as an operator on $L^2(\mathbf{R}^d)$. Actually, as a function de χ , $M^*(E)$ follows a “0-1 law”: it is 0 if $\text{supp } \chi \subset]0, 2\pi[$, and unitary if χ equals 0 near 0, and 1 near 2π . For the model case one has $M^*(E)v(x) = e^{-2i\pi E/h} e^{\pi\mu} v(xe^{2\pi\mu})$ when $\int \chi'(t) dt = 1$. We check the unitarity of $M^*(E)$ as follows: by the first equality (2.4)

$$\begin{aligned} M^*(E)M(E) &= L_{2\pi}^*(E) \frac{i}{h} [H, \chi] L_0(\cdot, E) L_0^*(E) \frac{i}{h} [H, \chi] L_{2\pi}(\cdot, E) \\ &= L_{2\pi}^*(E) \left(\int dE' L_0(\cdot, E') P_0(\cdot, E') L_0^*(E') \right) \frac{i}{h} [H, \chi] L_0(\cdot, E) L_0^*(E) \\ &\quad \left(\int dE'' L_{2\pi}(\cdot, E'') P_{2\pi}(\cdot, E'') L_{2\pi}^*(E'') \right) \frac{i}{h} [H, \chi] L_{2\pi}(\cdot, E) \end{aligned}$$

that is

$$\begin{aligned} M^*(E)M(E) &= L_{2\pi}^*(E) \int dE' L_0(\cdot, E') P_0(\cdot, E') (L_0^*(E') \frac{i}{h} [H, \chi] L_0(\cdot, E)) L_0^*(E) \\ &\quad \int dE'' L_{2\pi}(\cdot, E'') P_{2\pi}(\cdot, E'') (L_{2\pi}^*(E'') \frac{i}{h} [H, \chi] L_{2\pi}(\cdot, E)) \end{aligned}$$

Expanding the kernels by stationary phase, we take in account (2.3) to estimate the contribution of $L_0^*(E') \frac{i}{h} [H, \chi] L_0(\cdot, E)$ and $L_{2\pi}^*(E'') \frac{i}{h} [H, \chi] L_{2\pi}(\cdot, E)$ and use eventually the second equality (2.4). So $M^*(E)M(E) = \text{Id}$, and similarly $M(E)M^*(E) = \text{Id}$. We then check the structure of $M^*(E)$ using BNF, and find that

$$M^*(E)v(x, h) = \int dt \int \mathcal{M}^*(t, x, z) v(z) dz$$

with the kernel $\mathcal{M}^*(t, x, z)$ of the form

$$\mathcal{M}^*(t, x, z) = \chi'(t) \int e^{i\Psi_2(t, x, z, \eta')/h} m(t, x, z, \eta'; h) d\eta'$$

which is in BNF, and where the integral is independent of t . In fact $M^*(E) = e^{iR^w(x, hD_x; E, h)/h}$, where R is h -PDO in BNF, self-adjoint for real E .

6) *End of the proof.*

Reducing the spectral problem for $H^w(x, hD_x; h)$ through a Grushin operator as in [SjZw], [FaLoRo], we consider the approximate kernel of $M^*(E) - \text{Id}_{L^2(\mathbf{R}^d)}$ for complex values of E . In particular we know that for real E , $M^*(E)$ is (microlocally) unitary with absolutely continuous spectrum, while 1 belongs to its discrete spectrum for some complex values of E , which are precisely the resonances. This requires first to take suitable analytic extensions with respect to E of Poisson and monodromy operators. Since $M^*(E)$ is in BNF (formal) eigenfunctions of $M^*(E)$ are the homogeneous polynomials $f_\alpha(x)$ (of degree depending on the accuracy of BNF, and that we can take of order $h^{-\delta'}$, for some $0 < \delta' < 1$). This gives the “transverse” quantum numbers. The longitudinal quantum number m is found by taking also the phase of $M^*(E)f_\alpha$ equal to $2k\pi$, $k \in \mathbf{Z}$, of order also depending on the accuracy of BNF. Thus the Theorem is proved.

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